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A saddlepoint approximation for testing exponentiality against some increasing failure rate alternatives

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Abstract

In this article we discuss uniformly most powerful unbiased tests for testing exponentiality against a specific class of two-parameter exponential models with increasing failure rate. We show that the optimal test statistic for this problem admits an alternative representation in terms of a spacings statistic. Using the conditional saddlepoint approximation proposed by Gatto and Jammalamadaka (J. Amer. Statist. Assoc. 94 (1999) 533), we provide highly accurate approximations for the significance values. The test procedure is illustrated with two practical examples from reliability and survival analysis. We also obtain the asymptotic distribution of the test statistic under a sequence of converging alternatives, which allows for the computation of asymptotic relative efficiency. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Increasing failure rate (IFR) and decreasing failure rate (DFR) distributions play a very important role in reliability and survival studies. These two classes of distributions characterize life durations of physical units that deteriorate (for IFR) or that improve in reliability (for DFR) with time. The failure rate or hazard function $h(x)$ of an absolutely continuous random variable $X > 0$ (representing the life duration of a physical unit) is defined by $h(x) dx = P[X \in (x, x + dx) | X > x]$. It can be seen that IFR (respectively DFR) distributions on \mathbb{R}_+ are characterized by the fact minus the logarithm of the density function is a convex (respectively concave) function, see e.g. Barlow and Proschan

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(1981). We consider probability models with density function of the form

$$f(x|\theta) = c^{-1}(\theta) \exp\{-\theta_1 x - \theta_2 S(x)\}, \quad x \geq 0, \quad (1)$$

where S is a convex or a concave function, and $\theta = (\theta_1, \theta_2)$. These are two-parameter exponential models with parameters θ_1 and θ_2 , and were considered recently by del Castillo and Puig (1999a, b). It is important to note that because convexity is invariant under linear transforms, all distributions in these models are IFR or DFR, depending on whether S is convex or not. There are yet only three possible choices for the function S that lead to the model being scale invariant, i.e. σX remains in the same class of distributions, $\forall \sigma > 0$, and we focus here on the particular choice $S(x) = x^q$. We refer to alternatives $S(x) = x^q$ with $q > 1$ as Type I alternatives, and to alternatives with $-1 < q < 0$ as Type II alternatives. Both Types I and II alternatives represent IFR statistical models.

We now focus on the problem of testing exponentiality with alternatives of the types considered above. By the classical theory of uniformly most powerful unbiased (UMPU) tests for multiparameter exponential families (see e.g. Lehmann, 1994), tests for the null hypothesis $H_0 : \theta_2 = 0$ (i.e. for exponentiality) against the one-sided alternative $H_1 : \theta_2 > 0$ of size ε are characterized by rejection regions of the form $\{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n S(x_i) < K(\sum_{i=1}^n x_i)\}$, K being a function so that, under H_0 , $P[\sum_{i=1}^n S(X_i) < K(\sum_{i=1}^n X_i)] = \varepsilon$, for all values of the (nuisance) parameter θ_1 of the exponential distribution, and where X_1, \dots, X_n are independent random observations from the underlying model. del Castillo and Puig (1999a) show in particular that for alternatives of Types I and II, this general form of the rejection region of the UMPU test simplifies to

$$\{x \in \mathbb{R}_+^n \mid \bar{s}/\bar{x}^q < t_\varepsilon\}, \quad \text{where } \bar{s} = n^{-1} \sum_{i=1}^n S(x_i), \quad \bar{x} = n^{-1} \sum_{i=1}^n x_i,$$

and where t_ε is the value for which the rejection region of the test has size ε . Although this exact test is UMPU, the small sample distribution of the test statistic is not available. For both Types I and II alternatives, we show that by reexpressing the problem in terms of spacings, which are the gaps between the successive order statistics, it is possible to apply the conditional saddlepoint approximation for spacings statistics proposed by Gatto and Jammalamadaka (1999) in this setting. This saddlepoint approximation allows one to compute significance values or P -values with a relative error of the order of n^{-1} , or to determine critical values with a relative error of the order of $n^{-3/2}$. Tests based on spacings, which are the gaps between the successive values of an ordered uniform sample, have proved themselves useful in various statistical problems, including goodness-of-fit tests, see e.g. Pyke (1965) for a general review, and Doksum and Yandell (1984) in connection to reliability. For the special case of Type I model with $q=2$ (singly truncated normal alternatives) del Castillo and Puig (1999b) propose reexpressing the critical region of the test by means of a likelihood ratio statistic which enables one to use a specific saddlepoint approximation. By using the conditional approach by Gatto and Jammalamadaka (1999) together with a general spacings statistic representation, we obtain a more general approximation, covering the special case $q=2$ (which corresponds to the well-known Greenwood spacings statistic). Again by exploiting the spacings representation, we next derive the asymptotic null distribution of the test statistic for exponentiality, as well as its distribution under close alternatives.

The structure of this article is as follows. In Section 2, we describe the saddlepoint approximation for this particular setting. In Section 3, we give the asymptotic distributions of the test statistic for exponentiality under the null hypothesis as well as under a sequence of alternatives converging to the null hypothesis, which allows one to obtain Pitman’s asymptotic relative efficiency (ARE). The excellent numerical accuracy of the saddlepoint approximation is finally discussed in Section 4 and illustrated by means of two examples based on real data, one on air conditioning failures in aircraft and the other on remission times for leukemia.

2. Application of the conditional saddlepoint approximation

The saddlepoint method provides very accurate approximations for distribution or density functions of statistics, even in extreme situations (tails of the distribution, very small sample sizes). It was introduced into statistics by Daniels (1954) and some important general texts and reviews are: Barndorff-Nielsen and Cox (1989), Field and Ronchetti (1990), Field and Tingley (1997), Jensen (1995), and Kolassa (1994). To find the distribution of \bar{S}/\bar{X}^q , where $\bar{S} = n^{-1} \sum_{i=1}^n S(X_i) = n^{-1} \sum_{i=1}^n X_i^q$ and $\bar{X} = n^{-1} \sum_{i=1}^n X_i$, we apply the conditional saddlepoint approximation of a sample mean given another sample mean, proposed by Skovgaard (1987). To do this we exploit the conditional representation for exponential random variables given by (6) below.

The important steps for obtaining the saddlepoint approximation of the distribution function of the test statistic \bar{S}/\bar{X}^q for models of Types I and II are the following. Consider the cumulant generating function

$$K(\lambda; t) = -\lambda_1 t - \lambda_2 + \log \int_0^\infty \exp\{\lambda_1 x^q + (\lambda_2 - 1)x\} dx, \tag{2}$$

where $\lambda = (\lambda_1, \lambda_2)$ and $t > 0$ is the point at which we evaluate the distribution function.

Step 1: Find $\alpha = (\alpha_1, \alpha_2)$, the joint saddlepoint, implicitly defined by the equation

$$\frac{\partial}{\partial \lambda} K(\lambda; t) = 0.$$

Step 2: Define

$$K''(\lambda) = \frac{\partial^2}{\partial \lambda \partial \lambda^T} K(\lambda; t),$$

$$s = n^{1/2} \alpha_1 |\det(K''(\alpha))|^{1/2}, \quad r = n^{1/2} \operatorname{sgn}(\alpha_1) \{-2K(\alpha; t)\}^{1/2}$$

and

$$P_n(t) = \Phi(r) - \phi(r) \left\{ \frac{1}{s} - \frac{1}{r} \right\}, \tag{3}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and distribution functions.

Then, uniformly for t in sets where $(t - E[\bar{S}/\bar{X}^q]) = O(1)$, i.e. in large deviation regions as $n \rightarrow \infty$,

$$P \left[\frac{\bar{S}}{\bar{X}^q} < t \right] = P_n(t) \{1 + O(n^{-1})\}. \tag{4}$$

The error would also become $O(n^{-3/2})$ over regions where $(t - E[\bar{S}/\bar{X}^q]) = O(n^{-1/2})$, i.e. in normal deviation regions. The type of tail area probability approximation (3) is due to Lugannani and Rice (1980). Alternatively, the Barndorff-Nielsen tail area formula is given by

$$P_n^*(t) = \Phi \left(r + \frac{1}{r} \log \left\{ \frac{s}{r} \right\} \right), \quad (5)$$

and, by Lemma 2.1 in Jensen (1992), $P_n(t) = P_n^*(t)\{1 + O(n^{-1})\}$, uniformly in large deviation regions. The asymptotic term above would also become $O(n^{-3/2})$ in normal deviation regions.

Gatto and Jammalamadaka (1999) exploited a conditional representation of exponential random variables to obtain a saddlepoint approximation for spacings statistics, and their saddlepoint approximation is the same as the one given by Steps 1 and 2 above. The gaps between the successive values of an ordered sample are called the spacings. If U_1, \dots, U_{n-1} are independent and identically distributed (i.i.d.) random variables uniformly distributed on $[0, 1]$, and $0 \leq U_{(1)} \leq \dots \leq U_{(n-1)} \leq 1$ are their order statistics, we define the “uniform spacings” as

$$D_i = U_{(i)} - U_{(i-1)}, \quad i = 1, \dots, n,$$

where $U_{(0)} \stackrel{\text{def}}{=} 0$ and $U_{(n)} \stackrel{\text{def}}{=} 1$. If V_1, \dots, V_n are i.i.d. exponential random variables with distribution function $P[V_1 \leq v] = 1 - \exp\{-\xi v\}$, $v, \xi > 0$, then it is known that, $\forall \xi > 0$,

$$(nD_1, \dots, nD_n) \sim \left(\frac{V_1}{\bar{V}}, \dots, \frac{V_n}{\bar{V}} \right) \sim (V_1, \dots, V_n) \Big| \sum_{i=1}^n V_i = n, \quad (6)$$

where $\bar{V} = n^{-1} \sum_{i=1}^n V_i$. It follows that \bar{S}/\bar{X}^q is equivalent in distribution to a sum of spacings, each one raised to the q th power. This can also be reexpressed as a sum of i.i.d. exponential random variables each one raised to the q th power and then conditioned on their sum, which allows for the application of the conditional saddlepoint approximation for spacing statistics in the form provided by Gatto and Jammalamadaka (1999). In particular, we set $\beta = 0$ and $t_2 = 1$ as defined in Gatto and Jammalamadaka (1999, p. 534). The point of conditioning is indeed $t_2 = 1$ because we condition on $\bar{V} = 1$, and the solution of the “conditional saddlepoint equation” is the trivial value $\beta = 0$ because we are free to choose the parameter of the exponential distribution $\xi = 1$ here above, so that the expectation of \bar{V} is one and no exponential tilting is then required.

In order to have $c(\theta) < \infty$ in the two-parameters exponential model (1), it is necessary to consider the following restrictions on the parametric space. Let us denote the parametric space as D , $\theta = (\theta_1, \theta_2) \in D$, the interior of D as Θ , and the boundary of D as $\Theta_0 = D \setminus \Theta$. For Type I, alternatives

$$\Theta = \{(\theta_1, \theta_2) \mid \theta_1 \in \mathbb{R}, \theta_2 > 0\} \quad \text{and} \quad \Theta_0 = \{(\theta_1, \theta_2) \mid \theta_1 > 0, \theta_2 = 0\}. \quad (7)$$

For Type II, alternatives

$$\Theta = \{(\theta_1, \theta_2) \mid \theta_1 > 0, \theta_2 > 0\} \quad \text{and} \quad \Theta_0 = \{(\theta_1, \theta_2) \mid \theta_1 > 0, \theta_2 = 0\}. \quad (8)$$

We can deduce from (7) and (8) that the integral in the cumulant generating function (2) converges under the following conditions: for Type I, $\lambda_1 < 0$, $\lambda_2 \in \mathbb{R}$, or $\lambda_1 = 0$, $\lambda_2 < 1$; for Type II, $\lambda_1 < 0$, $\lambda_2 < 1$.

Remark 2.1. The integral appearing in the cumulant generating function can generally not be solved analytically. In the two examples of Section 4, we evaluated it numerically by a Simpson rule, after an appropriate change of variables transforming the bounds of integration to become 0 and 1.

Remark 2.2. The special case where $q = 2$ has been the central subject of the paper by del Castillo and Puig (1999b), who show that in this case the critical region of the test can be reexpressed as $W > k_\varepsilon$, where $W = 2[l(\hat{\theta}) - l(\tilde{\theta})]$ is the likelihood ratio test statistic, l is the log-likelihood function, and $\hat{\theta}$ and $\tilde{\theta}$ are the MLE under H_1 and H_0 . Hence, the saddlepoint approximation for the signed root likelihood ratio statistic $R = \text{sgn} \hat{\theta}_2 W^{1/2}$ in exponential families can be used here (see e.g. Jensen, 1995, Section 5.2). Up to a second-order asymptotic accuracy, $R^* = R + R^{-1} \log\{S/R\}$ is approximately standard normal, where $S = \hat{\theta}_2 |\det\{I(\hat{\theta})/I_2(\tilde{\theta}_2)\}|^{1/2}$, $I(\theta) = -E_\theta[\partial^2/(\partial\eta\partial\eta^T) \log f(X_1 | \eta)]_{\eta=\theta}$ is the Fisher information matrix, and $I_2(\theta_2) = -E_{\theta_2}[\partial^2/\partial\eta^2 \log f(X_1 | (0, \eta))]_{\eta=\theta_2}$ is the scalar Fisher information under H_0 .

The Greenwood test statistic $n^{-1} \sum_{i=1}^n (nD_i)^2$ for the null hypothesis of uniformity corresponds to our exponential model with the singly truncated normal model alternative. In connection with this, Gatto and Jammalamadaka (1999) provide an alternate analytical solution for the integral in (2) for $q = 2$, yielding

$$K(\lambda, t) = \begin{cases} -\lambda_1 t_1 - \lambda_2 - \frac{(\lambda_2 - 1)^2}{4\lambda_1} + \frac{1}{2} \log\{-\frac{\pi}{\lambda_1}\} + \log\{\Phi(\frac{\lambda_2 - 1}{\sqrt{-2\lambda_1}})\}, & \text{if } \lambda_1 < 0, \\ -\lambda_2 - \log\{1 - \lambda_2\} & \text{if } \lambda_1 = 0 \text{ and } \lambda_2 < 1, \end{cases}$$

which avoids numerical integration. We provide here a single computation of the saddlepoint approximation for the distribution of the Greenwood test statistic, i.e. of \bar{S}/\bar{X}^q when $q = 2$. We consider the very small sample size $n = 4$. In this case, as $\sqrt{n}(\bar{S}/\bar{X}^q - 2) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 4)$, the asymptotic approximation to the distribution of \bar{S}/\bar{X}^q is given by $\Phi(\sqrt{n}(t - 2)/2)$ (see Section 3). The saddlepoint approximation for the distribution of \bar{S}/\bar{X}^q at $t = 1.156$ is 0.107 when based on the Lugannani and Rice formula, 0.098 when based on the Barndorff-Nielsen formula, these values compared to the exact value of 0.100 (Burrows, 1979). From the asymptotic normal approximation, we obtain a probability close to the double of the exact one (i.e. the relative error is close to 100%).

To summarize, the saddlepoint approximation for the case $q = 2$ proposed by del Castillo and Puig (1999b) turns out to be just a special case of our saddlepoint approximation. We also provide the normal approximation (derived in the next section), in order to show the importance of the saddlepoint approximation, with small sample sizes.

Remark 2.3. In order to determine critical regions of size ε , the ε th quantile of \bar{S}/\bar{X}^q , t_ε , can be approximated by a one-step inversion of P_n^* , see Wang (1995) and Gatto (2001) for details. This one-step method leads to an approximation of t_ε with a relative error of the order $n^{-3/2}$, and has shown to be numerically very accurate with the likelihood ratio goodness-of-fit test by Gatto (2001).

Remark 2.4. For Type II models, i.e. with $-1 < q < 0$, the derivatives K''_{11} and K''_{12} must be computed numerically, as it is not possible to exchange the order of derivation and integration. The

numerical differentiation does however not add significant inaccuracies, as we can see in Example 3, with $q = -1/2$. In Example 2, with $q = 2$ where these derivatives can be found analytically, very similar numerical results have been obtained with both the analytical and the numerical differentiations.

3. Asymptotic distributions under close alternatives

Our test of exponentiality involves the null hypothesis

$$H_0 : F(x) = F(x | \theta_1, 0) = 1 - \exp\{-\theta_1 x\},$$

and the alternative hypothesis

$$H_1 : F(x) = F(x | \theta_1, \theta_2) = c^{-1}(\theta) \int_0^x \exp\{-\theta_1 t - \theta_2 t^q\} dt, \quad \theta_2 > 0.$$

For a consistent test, the power at any fixed alternative tends to one as $n \rightarrow \infty$. Hence, asymptotic comparisons of such tests can only be made for sequences of alternatives converging towards the null as $n \rightarrow \infty$ at a sufficiently fast rate. If however this sequence of alternatives converges to the null too fast, then the test would not be able to distinguish between the null and the asymptotic sequence of alternatives, and the power would not be any bigger than the size. Typically, there is a specific convergence rate for the sequence of alternatives at which the asymptotic power will be smaller than one and larger than the size. At that specific rate, the test will of course not be consistent, but it will be possible to compute a non-trivial asymptotic power. This turns out to be useful for comparison purposes. In this context, Pitman's ARE can be defined as the inverse of the sample sizes required by two competing tests to attain a specific power between the size and one, for such converging sequences of alternatives (see e.g. Serfling, 1980, Section 10.2).

In what follows we derive the asymptotic distributions of our test statistic for exponentiality under appropriate asymptotically converging alternatives and, as a special case, we can obtain the asymptotic distribution under the null hypothesis. Although we provide this asymptotic approximation, we recall that it is known that spacings statistics converge rather slowly to normality, and therefore the saddlepoint approximation provided in Section 2 is a more reliable method for probability computations under the null hypothesis.

We take thus $H_{1n} : F(x) = F(x | \theta_1, \theta_{2n}) \xrightarrow{n \rightarrow \infty} F(x | \theta_1, 0)$, $\forall x, \theta_1 > 0$, which is the case when the sequence $\{\theta_{2n}\}$ converges to 0. From now on we write $F_0(x) = F(x | \theta_1, 0)$ and $F_{1n}(x) = F(x | \theta_1, \theta_{2n})$. Let us now define the random variables $D_i = nX_i/\bar{X}$, $i = 1, \dots, n$. Our test statistic for the null hypothesis of exponentiality based on the original sample X_1, \dots, X_n is now equal to a goodness-of-fit spacings statistic for the null hypothesis of uniformity, $\bar{S}/\bar{X}^q = n^{-1} \sum_{i=1}^n (nD_i)^q$, because the first equivalence in (6) allows for this "dual" interpretation. (In the dual goodness-of-fit problem, the ordered sample can be constructed by $U_{(1)} = D_1$, $U_{(2)} = D_1 + D_2, \dots, U_{(n-1)} = D_1 + \dots + D_{n-1}$.) The particular deviation from exponentiality written above as H_{1n} can now be reexpressed as a particular deviation from uniformity, $H_{1n} : G(u) = F_{1n}(F_0^{(-1)}(u))$, since $P_{H_{1n}}[F_0(X_1) \leq u] = F_{1n}(F_0^{(-1)}(u))$. Moreover, by defining

$$L_n(u) = n^{1/4} \{F_{1n}(F_0^{(-1)}(u)) - u\}, \quad 0 \leq u \leq 1, \quad (9)$$

we can rewrite the latter sequence of alternatives as

$$H_{1n} : G(u) = u + n^{-1/4}L_n(u).$$

We consider the following regularity conditions as given in Rao and Sethuraman (1975, p. 309): $L_n(0)=L_n(1)=0$, L_n continuously differentiable, i.e. with $l_n=L'_n$ continuous, $L_n \xrightarrow{n \rightarrow \infty} L$ uniformly, and L twice continuously differentiable, with $l=L'$. It was established by Rao and Sethuraman (1975, p. 312) that statistics based symmetrically on spacings cannot discriminate alternatives H_{1n} at rates $n^{-\delta}$, $\delta > 1/4$, and that the rate $n^{-1/4}$ is appropriate to obtain the asymptotic power over regions of modest power. With simple algebra we find from (9),

$$\begin{aligned} l_n(u) &= n^{1/4}[\theta_1^{-1}c^{-1}(\theta_1, \theta_{2n}) \exp\{-\theta_{2n}[-\theta_1^{-1} \log(1-u)]^q\} - 1] \\ &= n^{1/4}[\theta_{2n}\{\theta_1^{-q}\Gamma(q+1) - (-\theta_1^{-1} \log\{1-u\})^q\} + O(\theta_{2n}^2)]. \end{aligned}$$

This shows that we need $\theta_{2n} = n^{-1/4}\gamma + O(n^{-r})$, for $\gamma > 0$ and $r > 1/4$. With this choice we have $l_n(u) \xrightarrow{n \rightarrow \infty} l(u) = \gamma\{\theta_1^{-q}\Gamma(q+1) - (-\theta_1^{-1} \log\{1-u\})^q\}$. Under this sequence of alternatives H_{1n} , $n^{-1} \sum_{i=1}^n (nD_i)^q$ is then approximately normal with expectation

$$\mu_n = \Gamma(q+1) + \frac{1}{2\sqrt{n}} q(q-1)\Gamma(q+1) \int_0^1 l^2(u) du$$

and variance

$$\sigma_n^2 = \frac{1}{n} \{ \Gamma(2q+1) - (q^2+1)\Gamma^2(q+1) \}.$$

In particular, under H_0 , $l=0$ so that $\int_0^1 l^2(u) du = 0$. In this case the expected value of the test statistic under H_0 is $\mu_n = \Gamma(q+1)$, and the variance σ_n^2 remains fixed. As it could be expected, the expectation under any sequence of alternatives (and hence the local power) depends on the nuisance parameter θ_1 .

4. Applications

In this section, we consider applications of the proposed test and saddlepoint approximation to two real data examples. We show the high numerical accuracy of the conditional saddlepoint approximation with $q=3$ and $-1/2$, and with sample sizes $n=21$ and 29 , respectively.

Example 1 (*Remission times for leukemia*). This example corresponds to the remission times in weeks of $n=21$ patients with acute leukemia, proposed by Freireich et al. (1963), which is discussed in our setting by del Castillo and Puig (1999a). They show by a Weibull probability plot of these data (because the Weibull model has been used for these data by another author) that model (1) with $S(x) = x^q$ and $q=3$ (Type I) gives a good fit to the data. They also motivated the model with $q=3$ by comparing the associated hazard function h (a quadratic function) with the empirical hazard function. The likelihood equations of the model with $q=3$ have a unique solution, as they

Table 1
Distributions of \bar{S}/\bar{X}^q with $q = 3$ and $n = 21$, with P -values of the remission times for leukemia data

t	$P_E[\bar{S}/\bar{X}^q < t]$	$P_{LR}[\bar{S}/\bar{X}^q < t]$	$P_{BN}[\bar{S}/\bar{X}^q < t]$	$RAE_{LR}(t)$	$RAE_{BN}(t)$
1.800	0.001	0.001	0.001	0.193	0.174
2.000	0.003	0.004	0.004	0.072	0.056
2.200	0.011	0.012	0.012	0.041	0.025
2.400	0.027	0.028	0.027	0.011	0.005
2.600	0.052	0.053	0.052	0.023	0.007
2.800	0.086	0.089	0.087	0.028	0.011
2.904	0.108	0.110	0.108	0.024	0.007
3.000	0.129	0.132	0.130	0.027	0.009
3.200	0.176	0.183	0.179	0.037	0.019
3.400	0.228	0.237	0.233	0.041	0.022
3.600	0.280	0.295	0.289	0.052	0.031
3.800	0.333	0.353	0.346	0.060	0.038
4.000	0.385	0.411	0.402	0.068	0.045
4.200	0.432	0.466	0.455	0.077	0.052
4.400	0.480	0.515	0.503	0.074	0.048

Exact left tail probability (P_E), Lugannani and Rice (P_{LR}) and Barndorff–Nielsen (P_{BN}) approximations, absolute relative error of Lugannani and Rice ($RAE_{LR} = |P_E - P_{LR}|/P_E$) and Barndorff–Nielsen ($RAE_{BN} = |P_E - P_{BN}|/P_E$).

satisfy the condition given in Section 1, and the MLE are $\hat{\theta}_1 = 0.0313$ and $\hat{\theta}_2 = 0.0001$. We now show the accuracy of the saddlepoint approximation in this example. The integral in the cumulant generating function (2) is evaluated by a quadratic interpolation of the integrand (Simpson rule), after a change of variable which transforms the domain of integration from $(0, \infty)$ onto $(0, 1)$. The interval $(0, 1)$ is divided into m similar parts and we need evaluating the integrand the at the $m + 1$ extremities; this integration will generate an error of the order $O(m^{-4})$. We selected $m=1000$, leading to a negligible error of the order of 10^{-12} . In Table 1, P_E denotes the cumulative probabilities of the distribution of \bar{S}/\bar{X}^q , $q = 3$, obtained by 10^5 Monte Carlo samplings, which will be referred as “exact” probabilities (as they approximate the exact probabilities with negligible errors). P_{LR} denotes the probabilities obtained by the conditional saddlepoint approximation, in particular with the Lugannani and Rice method (3). P_{BN} denotes the probabilities obtained by the asymptotically equivalent Barndorff–Nielsen formula (5). The relative absolute error of both approximations is given by $RAE_{LR}(t) = |P_E[\bar{S}/\bar{X}^q < t] - P_{LR}[\bar{S}/\bar{X}^q < t]|/P_E[\bar{S}/\bar{X}^q < t]$, for the Lugannani and Rice formula, and $RAE_{BN}(t)$ is defined in a similar way. We can see that the saddlepoint approximation is extremely accurate, for very small tail probabilities also. Also, there is apparently not a significant difference between both Lugannani and Rice and Barndorff–Nielsen formulas. The exact P -value is also given in Table 1, namely 0.108, which is very well approximated by both the saddlepoint formulas. We can remark that, in this example, there is not a very strong evidence against the exponential model $\theta_2 = 0$.

Example 2 (Air-conditioning failures). We consider $n=29$ times in operating hours between successive failures of the air conditioning equipment of a aircraft. This example was proposed by Proschan

Table 2
Distributions of \bar{S}/\bar{X}^q with $q = -1/2$ and $n = 29$, with P -values of the air-conditioning failures data

t	$P_E[\bar{S}/\bar{X}^q < t]$	$P_{LR}[\bar{S}/\bar{X}^q < t]$	$P_{BN}[\bar{S}/\bar{X}^q < t]$	$RAE_{LR}(t)$	$RAE_{BN}(t)$
1.200	0.002	0.002	0.002	0.004	0.003
1.240	0.008	0.008	0.008	0.033	0.027
1.280	0.021	0.022	0.022	0.035	0.029
1.287	0.024	0.025	0.025	0.033	0.027
1.320	0.046	0.046	0.046	0.015	0.009
1.360	0.082	0.083	0.083	0.021	0.016
1.400	0.130	0.133	0.132	0.020	0.015
1.440	0.188	0.192	0.191	0.022	0.016
1.480	0.252	0.259	0.258	0.029	0.023
1.520	0.317	0.331	0.329	0.045	0.039
1.560	0.382	0.406	0.403	0.063	0.056
1.600	0.446	0.493	0.488	0.105	0.093

Exact left tail probability (P_E), Lugannani and Rice (P_{LR}) and Barndorff-Nielsen (P_{BN}) approximations, absolute relative error of Lugannani and Rice ($RAE_{LR} = |P_E - P_{LR}|/P_E$) and Barndorff-Nielsen ($RAE_{BN} = |P_E - P_{BN}|/P_E$).

(1963). Various other authors have analyzed these data and have often concluded that there was no strong evidence against the hypothesis of exponentiality. As an IFR alternative would be consistent with the nature of these data, del Castillo and Puig (1999a) considered the IFR model (1) with $q = -1/2$ (Type II). As in the previous example, the integral in the cumulant generating function (2) is evaluated by Simpson rule, and in Table 2, P_E denotes the cumulative probabilities of the distribution of \bar{S}/\bar{X}^q , $q = -1/2$, obtained by 10^5 Monte Carlo samplings, P_{LR} and P_{BN} denote the probabilities obtained by the Lugannani and Rice method (3) and by the asymptotically equivalent Barndorff-Nielsen formula (5). The relative absolute error of both approximations are again given by $RAE_{LR}(t) = |P_E[\bar{S}/\bar{X}^q < t] - P_{LR}[\bar{S}/\bar{X}^q < t]|/P_E[\bar{S}/\bar{X}^q < t]$, for the Lugannani and Rice formula, and $RAE_{BN}(t)$ is defined in a similar way. Here also, the saddlepoint approximation is very accurate, for very small tail probabilities also. The exact P -value corresponding to our sample is 0.024, as given in Table 2. Note that it differs from the Monte Carlo P -value of 0.027 provided by del Castillo and Puig (1999a), on the basis of 10^4 simulations. Both saddlepoint formulas provide very good approximations: 0.025. It appears here that the saddlepoint approximation is more reliable than the Monte Carlo computation with 10^4 simulations by del Castillo and Puig (1999a). In Fig. 1, we give a graphical representation of both the absolute errors regarding the saddlepoint formulas in the upper graph, and the relative absolute errors, in the lower figure. The absolute error for the Lugannani and Rice approximation is defined by $AE_{LR}(t) = |P_E[\bar{S}/\bar{X}^q < t] - P_{LR}[\bar{S}/\bar{X}^q < t]|$, and a similar definition holds for the Barndorff-Nielsen approximation. The relative absolute error is the same as the one defined in Table 1. For the saddlepoint approximation, K''_{12} and K''_{22} must be computed numerically, as we cannot differentiate inside the integral sign, see Remark 2.4. This was done on the basis of an elementary quadratic interpolation formula and, although numerical differentiation is sometimes not an accurate method, in this setting the resulting saddlepoint approximations turn out to be accurate. As in del Castillo and Puig (1999a), the conclusion of this

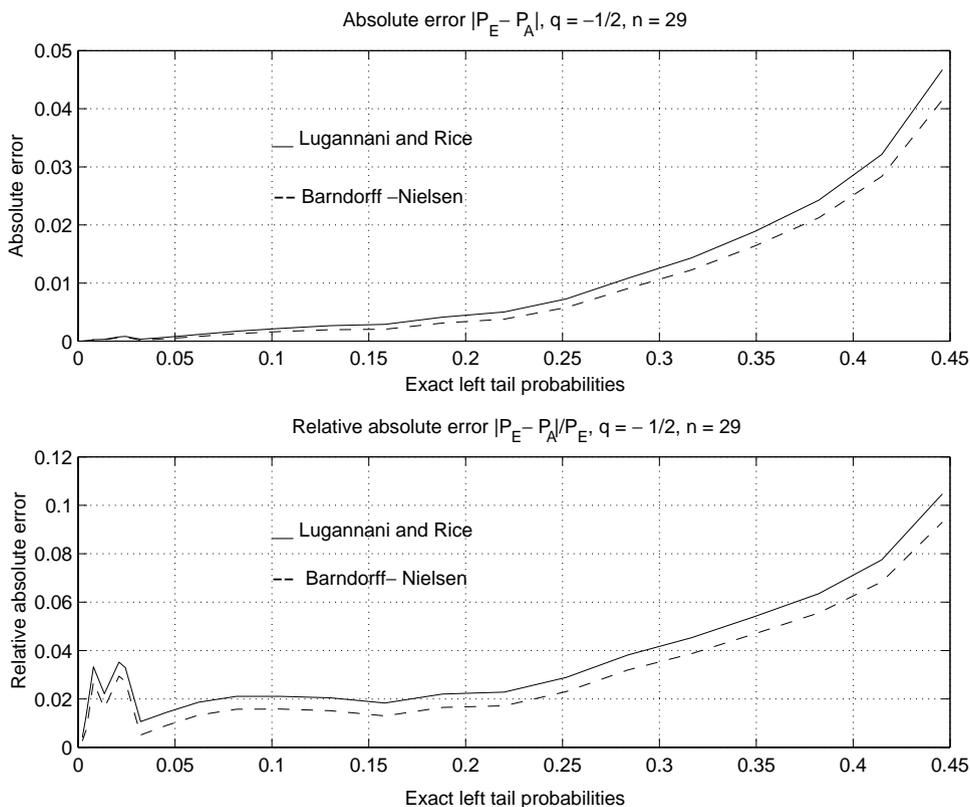


Fig. 1. Saddlepoint approximation for the distribution of \bar{S}/\bar{X}^q with $q = -1/2$ and $n = 29$. Upper figure: absolute error $|P_E - P_A|$. Lower figure: relative absolute error $|P_E - P_A|/P_E$. P_E : exact left tail probabilities. P_A : approximated left tail probabilities. Solid line: Lugannani and Rice approximation. Dashed line: Barndorff-Nielsen approximation.

example is that there is some evidence against the exponential model ($\theta_2 = 0$), in favor of Type II IFR model.

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